Introduction to Modal and Temporal Logic

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History: Logic of Necessity and Possibility

Classical logic is truth-functional: truth value of larger formula determined by truth value(s) of its subformula(e) via truth tables for \(\wedge, \vee, \neg, \text{ and } \rightarrow\).

Lewis 1920s: How to capture a non-truth-functional notion of “A Necessarily Implies B”? \((A \Rightarrow B)\)

Take \(\sim A\) to mean “it is impossible for \(A\) to be true and \(\neg A\) to be false”

Write \(P A\) for “\(A\) is possible” then:

\(-P A\) is “\(A\) is impossible”

\(-P \neg A\) is “not-\(A\) is impossible”

\(N A := \neg P \neg A\) “\(A\) is necessary”

\(A \sim B := N(A \rightarrow B) = \neg P(\neg A \rightarrow B) = \neg P(\neg A \vee B) = \neg P(A \wedge \neg B)\)

Modal Logic: “possibly true” and “necessarily true” are modes of truth

Preliminaries

Directed Graph \(\langle V, E \rangle\): where \(V = \{v_0, v_1, \cdots\}\) is a set of vertices and \(E = \{(s_1, t_1), (s_2, t_2), \cdots\}\) is a set of edges from source vertex \(s_i \in V\) to target vertex \(t_i \in V\) for \(i = 1, 2, \cdots\).

Cross Product: \(V \times V\) stands for \(\{(v, w) \mid v \in V, w \in V\}\) the set of all ordered pairs \((v, w)\) where \(v\) and \(w\) are from \(V\).

Directed Graph \(\langle V, E \rangle\): where \(V = \{v_0, v_1, \cdots\}\) is a set of vertices and \(E \subseteq V \times V\) is a binary relation over \(V\).

Iff: means if and only if.

Logic = Syntax and (Semantics or Calculus)

Syntax: formation rules for building formulae \(\varphi, \psi, \cdots\) for our logical language

Assumptions: a (usually) finite collection \(\Gamma\) of formulae

Semantics: \(\varphi\) is a logical consequence of \(\Gamma\) \((\Gamma \models \varphi)\)

Calculi: \(\varphi\) is derivable (purely syntactically) from \(\Gamma\) \((\Gamma \vdash \varphi)\)

Soundness: If \(\Gamma \vdash \varphi\) then \(\Gamma \models \varphi\)

Completeness: If \(\Gamma \models \varphi\) then \(\Gamma \vdash \varphi\)

Consistency: Both \(\Gamma \vdash \varphi\) and \(\Gamma \vdash \neg \varphi\) should not hold for any \(\varphi\)

Decidability: Is there an algorithm to tell whether or not \(\Gamma \models \varphi\) ?

Complexity: Time/space required by algorithm for deciding whether \(\Gamma \models \varphi\) ?
Syntax of Modal Logic

Atomic Formulae: \( p ::= p_0 \mid p_1 \mid p_2 \mid \cdots \) \((\text{Atm})\)

Formulae: \( \varphi ::= p \mid \neg \varphi \mid \langle \varphi \rangle \mid \langle \varphi \rangle \varphi \mid \varphi \lor \psi \mid \varphi \rightarrow \psi \) \((\text{Fml})\)

Examples: \( [p_0 \rightarrow p_2] \mid [p_0 \rightarrow p_1] \mid (p_1 \rightarrow p_2) \rightarrow ([p_1] \rightarrow [p_2]) \)

Variables: \( p, q, r \) stand for atomic formulae while \( \varphi, \psi \) possibly with subscripts stand for arbitrary formulae (including atomic ones)

Schema/Shapes: \( [\varphi] \rightarrow \varphi \quad [\varphi] \rightarrow [\varphi] \varphi \quad [\langle \varphi \rangle \rightarrow \psi] \rightarrow ([\varphi] \rightarrow [\psi]) \)

Kripke Semantics for Logical Consequence

Motivation: Give an intuitive meaning to syntactic symbols.

Examples: \( [p_0] \rightarrow p_0 \) is an instance of \( [\varphi] \rightarrow \varphi \) but \( [p_0] \rightarrow p_2 \) is not

Formula Length: number of logical symbols, excluding parentheses, where \( \text{length}(p_0) = \text{length}(p_1) = \cdots = 1 \)

Example: \( \text{length}([p_0] \rightarrow p_2) = 4 \)

Kripke Frame: directed graph \( \langle W, R \rangle \) where \( W \) is a non-empty set of points/worlds/vertices and \( R \subseteq W \times W \) is a binary relation over \( W \)

Valuation: on a Kripke frame \( \langle W, R \rangle \) is a map \( \vartheta : W \times \text{Atm} \rightarrow \{t, f\} \) telling us the truth value \( (t \text{ or } f) \) of every atomic formula at every point in \( W \)

Kripke Model: \( \langle W, R, \vartheta \rangle \) where \( \vartheta \) is a valuation on a Kripke frame \( \langle W, R \rangle \)

Example: If \( W = \{w_0, w_1, w_2\} \) and \( R = \{(w_0, w_1), (w_0, w_2)\} \) and \( \vartheta(w_1, p_3) = t \) then \( \langle W, R, \vartheta \rangle \) is a Kripke model as pictured below:

\[
\begin{array}{c|c}
R & \vartheta(w_0, p) \\
\hline
w_1 & f \text{ for all } p \in \text{Atm} \\
\hline
w_1 & f \text{ for all } p \neq p_3 \in \text{Atm} \\
\hline
w_2 & f \text{ for all } p \in \text{Atm} \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\langle \varphi \rangle & \vartheta(w_0, \langle \varphi \rangle) \\
\hline
\varphi & ? \\
\hline
\langle \varphi \rangle & ? \\
\hline
\end{array}
\]

Intuition: classical connectives behave as usual at a world \( \langle \neg \varphi \lor \psi \rangle \) (truth functional)

Kripke Semantics for Logical Consequence

Motivation: Define a meaning of \( \varphi \) is a logical consequence of \( \Gamma \) \( (\Gamma \models \varphi) \)

Goal: Prove some interesting properties of logical consequence.
Kripke Semantics for Logical Consequence

Given some model \( \langle W, R, \vartheta \rangle \) and some \( w \in W \), we compute the truth value of a non-atomic formula by recursion on its shape:

\[
\begin{align*}
\vartheta(w, \psi & ) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at some } v \in W \text{ with } wRv \\
  f & \text{otherwise} 
\end{cases} \\
\vartheta(w, \psi & ] \varphi) = \begin{cases} 
  t & \text{if } \vartheta(v, \varphi) = t \text{ at every } v \in W \text{ with } wRv \\
  f & \text{otherwise} 
\end{cases}
\end{align*}
\]

Example: If \( W = \{ w_0, w_1, w_2 \} \) and \( R = \{ (w_0, w_1), (w_0, w_2) \} \) and \( \vartheta(w_1, p_3) = t \) then \( \langle W, R, \vartheta \rangle \) is a Kripke model as pictured below:

```
R
w_0
\downarrow
w_2
\vartheta(w_0, \psi]\varphi) = t
\vartheta(w_0, p_3) = t
\vartheta(w_1, p_3) = f
\vartheta(w_1, \psi]\varphi) = t
\vartheta(w_1, \psi]p_1) = t
\vartheta(w_0, \psi]p_1) = f
```

Intuition: truth of modalities depends on underlying \( R \) (not truth functional)

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Classical (Two-Valued) Nature of Kripke Semantics

Lemma 1 For any Kripke model \( \langle W, R, \vartheta \rangle \), any \( w \in W \) and any formula \( \varphi \), either \( \vartheta(w, \varphi) = t \) or else \( \vartheta(w, \varphi) = f \).

Proof: Pick any Kripke model \( \langle W, R, \vartheta \rangle \), any \( w \in W \), and any formula \( \varphi \). Proceed by induction on the length \( l \) of \( \varphi \).

Base Case \( l = 1 \): If \( \varphi \) is an atomic formula \( \psi \), either \( \vartheta(w, \psi) = t \) or \( \vartheta(w, \psi) = f \) by definition of \( \vartheta \). So the lemma holds for all atomic formulae.

Ind. Hyp.: Lemma holds for all formulae of length less than some \( n > 0 \).

Induction Step: If \( \varphi \) is of length \( n \), then consider the shape of \( \varphi \).

\( \varphi = \psi \): If \( w \) has no \( R \)-successors, then \( \vartheta(w, \psi) = f \), and \( \vartheta(w, \psi) = t \) is impossible by its definition. Else pick any \( v \in W \) with \( wRv \). By IH, either \( \vartheta(v, \psi) = t \) or else \( \vartheta(v, \psi) = f \) since \( \psi \) is smaller than \( \varphi \). Either all \( R \)-successors of \( w \) make \( \psi \) false, or else at least one of them makes \( \psi \) true. Hence, either \( \vartheta(w, \psi) = f \) or else \( \vartheta(w, \psi) = t \).

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Semantic Forcing Relation \( \models \)

Let \( \mathcal{K} \) be the class of all Kripke models, and \( \mathcal{M} = \langle W, R, \vartheta \rangle \) a Kripke model.

Let \( \mathfrak{F} \) be the class of all Kripke frames, and let \( \mathfrak{G} \) be a Kripke frame.

Let \( \Gamma \) be a set of formulae, and \( \varphi \) a formula.

<table>
<thead>
<tr>
<th>Forces</th>
<th>We say</th>
<th>We Write</th>
<th>When</th>
<th>( \models \varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>In a world</td>
<td>( w ) forces ( \varphi )</td>
<td>( w \Vdash \varphi )</td>
<td>( \vartheta(w, \varphi) = t )</td>
<td>( \vartheta(w, \varphi) = f )</td>
</tr>
<tr>
<td>In a model</td>
<td>( \mathcal{M} ) forces ( \varphi )</td>
<td>( \mathcal{M} \models \varphi )</td>
<td>( \forall w \in W, W, w \Vdash \varphi )</td>
<td>( \exists w \in W, w \not\Vdash \varphi )</td>
</tr>
<tr>
<td>In a frame</td>
<td>( \mathfrak{G} ) forces ( \varphi )</td>
<td>( \mathfrak{G} \Vdash \varphi )</td>
<td>( \forall \vartheta, \langle \mathfrak{G}, \vartheta \rangle \Vdash \varphi )</td>
<td>( \exists \vartheta, \langle \mathfrak{G}, \vartheta \rangle \not\Vdash \varphi )</td>
</tr>
</tbody>
</table>

Classicality: \( \text{either } \mathfrak{G} \Vdash \varphi \text{ or else } \mathfrak{G} \not\Vdash \varphi \text{ holds for } \mathfrak{G} \in \{ \mathcal{M}, \mathfrak{G} \} \)

Exercise: Work out the negation of each fully e.g. \( \mathcal{M} \not\Vdash \varphi \) is \( \exists w \in W, w \Vdash \varphi \). Either \( w \Vdash \varphi \) or else \( w \not\Vdash \varphi \) holds (Lemma 1)

But this does not apply to all: e.g. \( \text{either } \mathcal{M} \Vdash \varphi \text{ or else } \mathcal{M} \not\Vdash \varphi \text{ is rarely true.} \)

\( W \Vdash \varphi \) meaning "every frame built out of given \( W \) forces \( \varphi \)" is not interesting
Various Consequence Relations

Let $\mathcal{K}$ be the class of all Kripke models, and $\mathcal{M} = (W, R, \theta)$ a Kripke model.

Let $\mathcal{F}$ be the class of all Kripke frames. and let $\mathcal{G}$ be a Kripke frame.

Let $\Gamma$ a set of formulae, and $\varphi$ a formula.

<table>
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<th>When</th>
<th>$\models_{\mathcal{F}} \varphi$</th>
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</thead>
<tbody>
<tr>
<td>in a world $w$ forces $\varphi$</td>
<td>$w \models_{\mathcal{F}} \varphi$</td>
<td>$\forall w \in W, w \models_{\mathcal{F}} \varphi$</td>
<td>$\exists w \in W, w \models_{\mathcal{F}} \varphi$</td>
<td></td>
</tr>
<tr>
<td>in a model $\mathcal{M}$ forces $\varphi$</td>
<td>$\mathcal{M} \models_{\mathcal{F}} \varphi$</td>
<td>$\forall \mathcal{M} \in \mathcal{K}, \mathcal{M} \models_{\mathcal{F}} \mathcal{M} \models_{\mathcal{F}} \varphi$</td>
<td>$\exists \mathcal{M} \in \mathcal{K}, \mathcal{M} \models_{\mathcal{F}} \mathcal{M} \models_{\mathcal{F}} \varphi$</td>
<td></td>
</tr>
<tr>
<td>in a frame $\mathcal{G}$ forces $\varphi$</td>
<td>$\mathcal{G} \models_{\mathcal{F}} \varphi$</td>
<td>$\forall \mathcal{G} \in \mathcal{F}, \mathcal{G} \models_{\mathcal{F}} \mathcal{G} \models_{\mathcal{F}} \varphi$</td>
<td>$\exists \mathcal{G} \in \mathcal{F}, \mathcal{G} \models_{\mathcal{F}} \mathcal{G} \models_{\mathcal{F}} \varphi$</td>
<td></td>
</tr>
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</table>

Let $\models_{\mathcal{F}} \Gamma$ stand for $\forall \varphi \in \Gamma, \models_{\mathcal{F}} \varphi$ ($\models_{\mathcal{F}} \{w, \mathcal{M}, \mathcal{G}\}$)

**World:** every world that forces $\Gamma$ also forces $\varphi$ $\forall w \in W, w \models_{\mathcal{F}} \Gamma \Rightarrow w \models_{\mathcal{F}} \varphi$

**Model:** every model that forces $\Gamma$ also forces $\varphi$ $\forall \mathcal{M} \in \mathcal{K}, \mathcal{M} \models_{\mathcal{F}} \Gamma \Rightarrow \mathcal{M} \models_{\mathcal{F}} \varphi$

**Frame:** every frame that forces $\Gamma$ also forces $\varphi$ $\forall \mathcal{G} \in \mathcal{F}, \mathcal{G} \models_{\mathcal{F}} \Gamma \Rightarrow \mathcal{G} \models_{\mathcal{F}} \varphi$

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Logical Consequence, Validity and Satisfiability

**Logical Consequence:** $\Gamma \models \varphi$ iff $\forall \mathcal{M} \in \mathcal{K}, \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi$

**Validity:** $\varphi$ is $\mathcal{K}$-valid iff $\emptyset \models \varphi$

**Satisfiability:** $\varphi$ is $\mathcal{K}$-satisfiable iff $\exists \mathcal{M} = (W, R, \theta) \in \mathcal{K}, \exists w \in W, w \models \varphi$

**Example:** $\{p_0\} \models \models \varphi$ means $\forall \mathcal{M} \models \varphi$.

For a contradiction, assume $\{\mathcal{G} \models \varphi\} \models \models \varphi$.

i.e. exists model $\mathcal{M} = (W, R, \theta)$ in $\mathcal{K}$, $\exists w \in W, w \models \varphi$

i.e. exists $w_0 \in W$ with $w_0 \models p_0$

i.e. exists $w_0 \in W$ with $w_0 \models \lnot p_0$

i.e. But $\mathcal{M} \models p_0$ means $\forall w \in W, w \models p_0$, hence $w_0 \models p_0$ (contradiction)

Exercise 1 All instances of $\varphi \to (\gamma \to \delta)$ are $\mathcal{K}$-valid.

Exercise 2 All instances of $(\varphi \to (\gamma \to \delta)) \to ((\varphi \to \gamma) \to (\varphi \to \delta))$ are $\mathcal{K}$-valid.

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Logical Consequence: Examples

Example 2 All instances of \( \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi) \) are \( \mathcal{K} \)-valid.

For a contradiction, assume there is some instance
\( \Box (\varphi_1 \rightarrow \psi_1) \rightarrow (\Box \varphi_1 \rightarrow \Box \psi_1) \) which is not \( \mathcal{K} \)-valid.

Therefore, there is some model \( \mathcal{M} = \langle W, R, \vartheta \rangle \) and some \( w \in W \) such that
\( w \not\models \Box (\varphi_1 \rightarrow \psi_1) \rightarrow (\Box \varphi_1 \rightarrow \Box \psi_1) \).

i.e. \( \vartheta(w, \Box (\varphi_1 \rightarrow \psi_1) \rightarrow (\Box \varphi_1 \rightarrow \Box \psi_1)) = f \)
i.e. \( w \not\models \Box (\varphi_1 \rightarrow \psi_1) \) and \( w \models \Box \varphi_1 \rightarrow \Box \psi_1 \)
i.e. \( w \models \Box (\varphi_1 \rightarrow \psi_1) \) and \( w \models \Box \varphi_1 \) and \( \Box \not\models \psi_1 \)
i.e. \( w \models \Box (\varphi_1 \rightarrow \psi_1) \) and \( \Box \models \varphi_1 \) and \( \Box \models \Box \psi_1 \)
i.e. \( w \models \varphi_1 \rightarrow \psi_1 \) and \( \varphi_1 \) and \( \Box \models \Box \psi_1 \)
i.e. \( w \models \varphi_1 \rightarrow \psi_1 \) and \( \varphi_1 \) and \( \Box \not\models \psi_1 \)

(contradiction)

Exercise 3 If \( \varphi \in \Gamma \) then \( \Gamma \models \varphi \)
(by definition of \( \models \))

Example 4 If \( \Gamma \models \varphi \) then \( \Gamma \models \Box \varphi \)
For a contradiction, assume \( \Gamma \models \varphi \) and \( \Gamma \not\models \Box \varphi \).
i.e. there is some model \( \mathcal{M} = \langle W, R, \vartheta \rangle \) such that \( \mathcal{M} \models \varphi \) and \( \mathcal{M} \not\models \Box \varphi \).
i.e. \( \mathcal{M} \models \varphi \) and \( \mathcal{M} \not\models \varphi \) and \( \mathcal{M} \not\models \Box \varphi \).
i.e. \( \mathcal{M} \models \varphi \) and \( \mathcal{M} \not\models \Box \varphi \).
But \( \Gamma \models \varphi \) means \( \forall \mathcal{M} \in \mathcal{K} . (\mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi) \), hence \( \forall \mathcal{M} \not\models \varphi \). Contradiction.

Exercise 3 If \( \Gamma \models \varphi \) and \( \Gamma \models \varphi \rightarrow \psi \) then \( \Gamma \models \psi \)

Logical Implication as Logical Consequence

Lemma 2 For any \( w \) in any model \( \langle W, R, \vartheta \rangle \), if \( w \models \{ \varphi, \varphi \rightarrow \psi \} \) then \( w \not\models \psi \)

Lemma 3 For any model \( \mathcal{M} \), if \( \mathcal{M} \models \{ \varphi, \varphi \rightarrow \psi \} \) then \( \mathcal{M} \models \psi \)

Lemma 4 If \( \Gamma \models \varphi \rightarrow \psi \) then \( \Gamma, \varphi \models \psi \)

Proof: Suppose \( \Gamma \models \varphi \rightarrow \psi \). Suppose \( \mathcal{M} \models \Gamma, \varphi \). Must show \( \mathcal{M} \models \psi \). But \( \mathcal{M} \models \Gamma \) implies \( \mathcal{M} \models \varphi \rightarrow \psi \), so \( \mathcal{M} \models \{ \varphi, \varphi \rightarrow \psi \} \). Lemma 3 gives \( \mathcal{M} \models \psi \).

Remark: Converse of Lemma 4 fails! e.g. We know \( p_0 \models \Box p_0 \). But \( \emptyset \models p_0 \rightarrow \Box p_0 \) is easily falsified in a model where \( w \models p_0 \), \( w R v \) and \( v \not\models \neg p_0 \).

Lemma 5 If \( \Gamma, \varphi \models \psi \) then there exists an \( n \) such that
\( \Gamma \models \{ \Box^0 \varphi \land \Box^1 \varphi \land \Box^2 \varphi \land \cdots \land \Box^n \varphi \} \rightarrow \psi \)
where \( \Box^0 \varphi = \varphi \) and \( \Box^n \varphi = \Box^{n-1} \varphi \)

Proof: For any \( m \in \{0, 1, \ldots, n\} \), if \( \Gamma \models \{ \Box^m \varphi \} \) then \( \Gamma \models \{ \Box^{m+1} \varphi \} \)

e.g. \( p_0 \models \Box^0 \varphi \) implies \( \emptyset \models \{ p_0 \land \Box p_0 \} \rightarrow \Box p_0 \) so \( n = 1 \) for this example

Logical \( = \) Syntax and Semantics

Atomic Formulae: \( p ::= p_0 | p_1 | p_2 | \cdots \) (Atn)

Formal Formulae: \( \varphi ::= p | \neg \varphi | (\Box \varphi) | \varphi \land \varphi | \varphi \lor \varphi | \varphi \rightarrow \varphi \) (Fml)

Kripke Frame: directed graph \( \langle W, R \rangle \) where \( W \) is a non-empty set of points/worlds/vertices and \( R \subseteq W \times W \) is a binary relation over \( W \).

Valuation on a Kripke frame \( \langle W, R, \vartheta \rangle \) is a map \( \vartheta : W \times Atm \rightarrow \{ t, f \} \) telling us the truth value \( t \) or \( f \) of every atomic formula at every point in \( W \).

Kripke Model: \( \langle W, R, \vartheta \rangle \) where \( \vartheta \) is a valuation on a Kripke frame \( \langle W, R \rangle \)
\( \Gamma \models \varphi \) iff \( \forall \mathcal{M} \in \mathcal{K} . \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi \)

Having defined \( \Gamma \models \varphi \), we can consider a logic to be a set of formulae:
\( \mathcal{K} = \{ \varphi \models \} = \{ \varphi \models \} \).
Lecture 2: Hilbert Calculi

Motivation: Define a notion of deducibility "\( \varphi \) is deducible from \( \Gamma \)"

Requirement: Purely syntax manipulation, no semantic concepts allowed.

Judgment: \( \Gamma \vdash \varphi \) where \( \Gamma \) is a finite set of assumptions (formulae)
- Read \( \Gamma \vdash \varphi \) as \( \varphi \) is derivable from assumptions \( \Gamma \)

Soundness: If \( \Gamma \vdash \varphi \) then \( \Gamma \vdash \varphi \)
- If \( \varphi \) is derivable from \( \Gamma \) then \( \varphi \) is a logical consequence of \( \Gamma \)

Completeness: If \( \varphi \) is derivable from \( \Gamma \) then \( \varphi \) is derivable from \( \Gamma \)
- If \( \varphi \) is a logical consequence of \( \Gamma \) then \( \varphi \) is derivable from \( \Gamma \)

Goal: Deducibility captures logical consequence via syntax manipulation.

Hilbert Derivability for Modal Logics

Assumptions: finite set of formulae accepted as derivable in one step
- (instantiation forbidden)

(Id) \( \vdash \varphi \in \Gamma \)

Axiom Schemata: Formula shapes, all of whose instances are accepted unquestionably as derivable in one step
- (listed shortly)

(Ax) \( \vdash \varphi \) is an instance of an axiom schema

Rules of Inference: allow us to extend derivations into longer derivations

Modus Ponens (MP) \( \Gamma \vdash \varphi, \Gamma \vdash \varphi \rightarrow \psi \) \( \Gamma \vdash \psi \)

Necessitation (Nec) \( \Gamma \vdash \varphi \) \( \Gamma \vdash [\varphi] \)

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Necessitation (Nec) \( \Gamma \vdash \varphi \) \( \Gamma \vdash [\varphi] \)

Derivation of \( \varphi_0 \) from assumptions \( \Gamma_0 \): is a finite tree of judgments with:
- a root node \( \Gamma_0 \vdash \varphi_0 \)
- only (Ax) judgment instances and (Id) as leaves
- and such that all parent judgments are obtained from their child judgments by instantiating a rule of inference
**Hilbert Calculus for Modal Logic K**

Axiom Schemata:

- **PC:** \( \varphi \rightarrow (\psi \rightarrow \varphi) \)
  
  - \( \neg \neg \varphi \rightarrow \varphi \)
  
  - \((\varphi \rightarrow (\psi \rightarrow \xi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \xi))\)

- **K:** \( [\![ \varphi \rightarrow \psi ]\!] \rightarrow ([\![ \varphi ]\!] \rightarrow [\![ \psi ]\!]) \)

**How used:** Create the leaves of a derivation via:

\((Ax) \quad \Gamma \vdash \varphi \quad \text{is an instance of an axiom schema}\)

\[ \varphi \land \psi \equiv \neg (\varphi \rightarrow \neg \psi) \]

\[ \varphi \lor \psi \equiv (\neg \varphi \rightarrow \psi) \]

\[ \varphi \leftrightarrow \psi \equiv (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi) \]

---

**Hilbert Derivations: Examples**

Let \( \Gamma_0 = \{p_0, p_0 \rightarrow p_1\} \) and \( \varphi_0 = [\![ p_1 ]\!] \). Usually omit braces.

Below is a derivation of \( [\![ p_1 ]\!] \) from \( \{p_0, p_0 \rightarrow p_1\} \).

\[
\begin{align*}
(p_0, p_0 &\rightarrow p_1 \vdash p_0) & \quad \text{(Id)} \\
(p_0, p_0 &\rightarrow p_1 \vdash p_0 \rightarrow p_1) & \quad \text{(Id)} \\
(p_0, p_0 &\rightarrow p_1 \vdash p_0 \rightarrow p_1) & \quad \text{(MP)} \\
p_0, p_0 &\rightarrow p_1 \vdash [\![ p_1 ]\!] \\
(p_0, p_0 &\rightarrow p_1 \vdash [\![ p_1 ]\!]) & \quad \text{(Nec)} \\
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash p_0 \\
\Gamma &\vdash [\![ p_0 ]\!] \\
\end{align*}
\]

\[
\begin{align*}
\Gamma := \{p_0, p_0 \rightarrow p_1\} \\
\varphi := p_1
\end{align*}
\]

---

**Hilbert Derivations: Examples**

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(p_0, p_0 &\rightarrow p_1 \vdash p_0 \rightarrow p_1) & \quad \text{(Id)} \\
p_0, p_0 &\rightarrow p_1 \vdash [\![ p_1 ]\!] \\
(p_0, p_0 &\rightarrow p_1 \vdash [\![ p_1 ]\!]) & \quad \text{(Nec)} \\
\end{align*}
\]

\[
\begin{align*}
\Gamma &\vdash \varphi \\
\Gamma &\vdash \psi \\
\end{align*}
\]

\[
\begin{align*}
\Gamma := \{p_0, p_0 \rightarrow p_1\} \\
\varphi := p_0 \\
\psi := p_1
\end{align*}
\]
Hilbert Derivations: Examples

Let $\Gamma_0 = \{p_0, p_0 \rightarrow p_1\}$ and $\varphi_0 = [\![p_1]\!]$. Usually omit braces.

Below is a derivation of $[\![p_1]\!]$ from $\{p_0, p_0 \rightarrow p_1\}$:

\[
\begin{array}{c}
\hline
p_0, p_0 \rightarrow p_1 \vdash p_0 \\
(\text{Id}) \\
p_0, p_0 \rightarrow p_1 \vdash p_0 \rightarrow p_1 \\
(\text{Id}) \\
p_0, p_0 \rightarrow p_1 \vdash p_1 \\
(\text{MP}) \\
p_0, p_0 \rightarrow p_1 \vdash [\![p_1]\!] \\
(\text{Nec}) \\
\hline
\end{array}
\]

\[
\begin{array}{c}
(\text{Id}) \quad \Gamma \vdash \varphi \in \Gamma \\
(\text{Id}) \quad \Gamma \vdash \varphi \in \Gamma \\
\hline
\end{array}
\]

Logic = Syntax and Calculus

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ \hspace{1cm} (Atn)

Formulæ: $\varphi ::= p \mid \neg \varphi \mid [\![\varphi]\!] \mid [\![\varphi \land \varphi]\!] \mid [\![\varphi \lor \varphi]\!] \mid [\![\varphi \rightarrow \varphi]\!]$ \hspace{1cm} (Fml)

Hilbert Calculus $K$: $[\![\varphi \rightarrow \psi]\!] \rightarrow ([\![\varphi]\!] \rightarrow [\![\psi]\!])$ \hspace{1cm} only modal axiom

\[
\begin{array}{c}
(\text{Id}) \quad \Gamma \vdash \varphi \in \Gamma \\
(\text{Ax}) \quad \Gamma \vdash \varphi \text{ is an instance of an axiom schema} \\
(\text{MP}) \quad \Gamma \vdash \varphi, \Gamma \vdash \varphi \rightarrow \psi \\
(\text{Nec}) \quad \Gamma \vdash \varphi \rightarrow [\![\varphi]\!] \\
\hline
\end{array}
\]

$\Gamma \vdash \varphi$ : iff there is a derivation of $\varphi$ from $\Gamma$ in $K$.

Having defined $\Gamma \vdash \varphi$, we can consider a logic to be a set of formulæ:

\[
K = \{\varphi \mid \emptyset \vdash \varphi\}
\]

$\varphi$ is a theorem of $K$ iff $\varphi \in K$ \hspace{1cm} i.e. if it is deducible from the empty set

A modal logic is called “normal” if it extends $K$ with extra modal axioms.

Soundness: all derivations are semantically correct

Theorem: \hspace{1cm} if $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$

Proof: By induction on the length $l$ of the derivation of $\Gamma \vdash \varphi$

$l = 0$: So $\Gamma \vdash \varphi$ because $\varphi \in \Gamma$. But $\mathcal{M} \models \varphi$ implies $\mathcal{M} \models \varphi$ for all $\varphi \in \Gamma$.

$l = 0$: So $\Gamma \vdash \varphi$ because $\varphi$ is an axiom schema instance. By Eg 1, Ex 1, Ex 2, Eg 2, we know $\emptyset \models \varphi$ for every axiom schema instance $\varphi$, hence $\Gamma \models \varphi$.

Ind. Hyp. : Theorem holds for all derivations of length less than some $k > 0$.

Ind. Step: Then $\Gamma \vdash \varphi$ has a derivation of length $k$. Bottom-most rule?

MP: So both $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$ are shorter than $k$. By IH $\Gamma \models \varphi \rightarrow \psi$ and $\Gamma \models \varphi$. But if $w \models \varphi \rightarrow \psi$ and $w \models \varphi$ then $w \models \psi$, hence $\Gamma \models \psi$.

Nec: Then we know that $\Gamma \vdash \varphi$ has length shorter than $k$. By IH we know $\Gamma \models \varphi$. But if $\Gamma \models \varphi$ then $\Gamma \vdash [\![\varphi]\!]$ by Eg 4.

Hilbert Derivations: Examples

Let $\Gamma = \{p_0, p_0 \rightarrow p_1\}$. Another derivation of $[\![p_1]\!]$ from $\{p_0, p_0 \rightarrow p_1\}$:

\[
\begin{array}{c}
\hline
p_0, p_0 \rightarrow p_1 \vdash p_0 \\
(\text{Id}) \\
p_0, p_0 \rightarrow p_1 \vdash p_0 \rightarrow p_1 \\
(\text{Id}) \\
p_0, p_0 \rightarrow p_1 \vdash p_1 \\
(\text{MP}) \\
p_0, p_0 \rightarrow p_1 \vdash [\![p_1]\!] \\
(\text{Nec}) \\
\hline
\end{array}
\]

\[
\begin{array}{c}
K: [\![\varphi \rightarrow \psi]\!] \rightarrow ([\![\varphi]\!] \rightarrow [\![\psi]\!]) \\
\varphi ::= p_0 \\
\psi ::= p_1 \\
\hline
\end{array}
\]
Completeness: all semantic consequences are derivable

Theorem: \[ \Gamma \models \varphi \text{ if } \Gamma \vdash \varphi \]

Proof Method: Prove contrapositive, if \( \Gamma \not\vdash \varphi \) then \( \Gamma \nvdash \varphi \)

Proof Plan: Assume \( \Gamma \not\vdash \varphi \) and show that there exists a \( \mathcal{K} \)-model \( \mathcal{M}_c = \langle W_c, R_c, \vartheta_c \rangle \) such that \( \mathcal{M}_c \models \Gamma \) and \( \mathcal{M}_c \not\models \varphi \), i.e. some world \( w_0 \in W_c \) such that \( w_0 \models \neg \varphi \)

Technique: is known as the canonical model construction

Set \( X \) is Maximal: \( \forall \varphi, \psi \in X \) or \( \neg \varphi \in X \)

Set \( X \) is Consistent: if both \( X \models \varphi \) and \( X \models \neg \varphi \) never hold, for any \( \varphi \)

Set \( X \) is Maximal-Consistent: if it is maximal and consistent.

Lindenbaum’s Construction of Maximal-Consistent Sets

Lemma 6 Every consistent \( \Gamma \) is extendable into a maximal-consistent \( X^* \supseteq \Gamma \).

Proof: Choose an enumeration \( \varphi_1, \varphi_2, \varphi_3, \ldots \) of the set of all formulae.

Stage 0: Let \( X_0 := \Gamma \)

Stage \( n > 0 \): \( X_n := \{ X_{n-1} \cup \{ \varphi_n \} \) if \( X_{n-1} \vdash \varphi_n \)
\( \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
The Canonical Model $\mathcal{M}_\Gamma = \langle W_c, R_c, \vartheta_c \rangle$

$W_c := \{ X^* | X^* \text{ is a maximal-consistent extension of } \Gamma \} \neq \emptyset$

$w R_c v \iff \{ \varphi \mid [\varphi] \in w \} \subseteq v$

$\vartheta_c(w, p) := \begin{cases} t \quad & \text{if } p \in w \\ f \quad & \text{otherwise} \end{cases}$

**Lemma 8** For every $w \in W_c$:

- $\lnot \varphi \in w \iff \varphi \notin w$

- $\varphi \lor \psi \in w \iff \varphi \in w \text{ and } \psi \in w$

- $\varphi \rightarrow \psi \in w \iff \varphi \notin w \text{ or } \psi \in w$

- $[\varphi] \in w \iff \forall v \in w, w R_c v \Rightarrow \varphi \in v$

- $\langle \rangle \in w \iff \exists v \in w, w R_c v \& \varphi \in v$

**Proof (ii):** Suppose $\varphi \land \psi \in w$ and $\varphi \notin w$. Then $\lnot \varphi \in w$.

Note $(\varphi \land \psi) \rightarrow \varphi \in w$ since $\emptyset \vdash (\varphi \land \psi) \rightarrow \varphi$ by PC (exercise)

Exists $l$ with $X_l \vdash \lnot \varphi$, and $X_l \vdash \varphi \land \psi$, and $X_l \vdash (\varphi \land \psi) \rightarrow \varphi$, all by (Id).

Then $X_l \vdash \varphi$ by (MP) Contradiction.

**Proof (i):** Suppose $\varphi \in w$ and $\varphi \notin w$. Then $\lnot \varphi \notin w$.

Note $(\varphi \rightarrow \lnot \varphi) \in w$ since $\emptyset \vdash (\varphi \rightarrow \lnot \varphi)$ (else can choose $\varphi_0 = \psi \rightarrow \varphi$ for some $\psi$)

i.e. exists $l$ such that $X_k \vdash \varphi$ and $X_k \vdash \varphi \land \psi \land \lnot \varphi$ and $X_k \vdash \psi$ by (id)

Then $X_k \vdash \lnot \varphi$ by (MP) Contradiction.
Truth Lemma

Lemma 9 For every \( \varphi \) and every \( w \in W_c: \vartheta_c(w, \varphi) = t \iff \varphi \in w. \)

Proof: Pick any \( \varphi \), any \( w \in W \). Proceed by induction on length \( l \) of \( \varphi \).

\( l = 0 \): So \( \varphi = p \) is atomic. Then, \( \vartheta_c(w, p) = t \iff p \in w \) by definition of \( \vartheta_c. \)

Ind. Hyp.: Lemma holds for all formulae with length \( l \) less than some \( n > 0 \)

Ind. Step: Assume \( l = n \) and proceed by cases on main connective

\( \varphi = [\psi] \): We have \( \vartheta_c(w, [\psi]) = t \)

iff \( \forall v \in W_c (wR_v \Rightarrow [\vartheta_c(v, \psi)] = t \) (by defn of valuations \( \vartheta_c) \)

iff \( \forall v \in W_c (wR_v \Rightarrow [\psi] \in v) \) (by IH)

iff \( [\psi] \in w \) by Lemma 8([]).

Completeness Proof

Corollary 1 \( (W_c, R_c, \vartheta_c) \vDash \Gamma \)

Proof: Since \( \Gamma \) is in every maximal-consistent set extending it, we must have \( \Gamma \subseteq w \) for all \( w \in W_c \). By Lemma 9, \( w \vDash \Gamma \), hence \( (W_c, R_c, \vartheta_c) \vDash \Gamma \)

Proof of Completeness: If \( \Gamma \not\vDash \varphi \)

Consider any ordering of formulae where \( \varphi \) is the first formula and the associated maximal-consistent extension \( X^* \). Since \( \Gamma \not\vDash \varphi \) we must have \( \neg \varphi \in X^* \). This particular set appears as some world \( w_0 \in W_c \) (say).

Hence there exists at least one world where \( \neg \varphi \in w_0 \). By Lemma 9 \( w_0 \vDash \neg \varphi \) i.e. \( \vartheta_c(w_0, \neg \varphi) \). By Corollary 1, we know \( \vartheta_c \vDash \neg \varphi \). Since the canonical model is a Kripke model, we have \( \Gamma \not\vDash \varphi \). (i.e. not \( \forall w \in \mathcal{K} . \vartheta_c(w, \varphi) \).

Completeness: By contraposition, if \( \Gamma \vDash \varphi \) then \( \Gamma \vDash \varphi \).

Notes

Completeness shows that \( \emptyset /\vDash \varphi \) implies \( \mathcal{M} \vDash \varphi \) i.e. \( \mathcal{M} \vDash \varphi \) implies \( \emptyset \vDash \varphi \)

How do we know that \( \emptyset \vDash \varphi \) implies \( \mathcal{M} \vDash \varphi \)?

Because the canonical frame is a Kripke frame by its definition.

Later we shall see that the canonical model is not always sound for \( \vdash \): that is we can have \( \emptyset \vdash \varphi \) but \( \mathcal{M} \not\vDash \varphi \).

Other books also use the notation \( \Gamma \vdash \varphi \), but they do not always use the same meaning e.g. Goldblatt uses \( \Gamma \vdash \varphi \) to mean “exists finite subset \( X \) of \( \Gamma \) such that \( \vdash (\Lambda \{ \psi \mid \psi \in X \}) \rightarrow \varphi \). For Goldblatt, the deduction theorem holds:

\( \Gamma, \varphi \vdash \psi \) iff \( \Gamma \vdash \varphi \rightarrow \psi \) since his deductions are totally local

For us, it takes the restricted form below by the fact that \( \models \) and \( \vdash \) are the same by soundness and completeness.

\( \Gamma, \varphi \vdash \psi \) iff \( \exists \Gamma, \Gamma \vdash ([\emptyset] \varphi \land [1] \varphi \land \ldots \land [n] \varphi) \rightarrow \psi \) ours is global

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Lecture 3: Logic = Syntax and (Semantics or Calculus)

\( \Gamma \vDash \varphi \): semantic consequence in class of Kripke models \( \mathcal{K} \)

\( \Gamma \vdash \varphi \): deducibility in Hilbert calculus \( \mathcal{K} \)

Soundness: if \( \Gamma \vdash \varphi \) then \( \Gamma \vDash \varphi \)

Completeness: if \( \Gamma \vdash \varphi \) then \( \mathcal{M} \not\vDash \varphi \) and \( \mathcal{M} \in \mathcal{K} \).

\( \mathcal{K} = \{ \varphi \mid \emptyset \vdash \varphi \} \) the validities of Kripke frames \( \mathcal{K} \)

\( \mathcal{K} = \{ \varphi \mid \emptyset \vdash \varphi \} \) the theorems of Hilbert calculus \( \mathcal{K} \)

Theorem 1 \( \mathcal{K} = \mathcal{K} \)

The presence of \( R \) makes modal logics non-truth-functionality.

But Kripke models put no conditions on \( R \).

So what happens if we put conditions on \( R \)?
Valid Shapes and Frame Conditions

A binary relation $R$ is reflexive if $\forall w \in W, w Rw$.

A frame $\langle W, R \rangle$ or model $\langle W, R, \vartheta \rangle$ is reflexive if $R$ is reflexive.

The shape $[\![ \varphi \implies \varphi ]\!]$ is called $T$.

A frame $\langle W, R \rangle$ validates a shape iff it forces all instances of that shape.

i.e. for all instances $\psi$ of the shape and all valuations $\vartheta$ we have $\langle W, R, \vartheta \rangle \models \psi$.

Lemma 10 A frame $\langle W, R \rangle$ validates $T$ iff $R$ is reflexive.

Intuition: the shape $T$ captures or corresponds to reflexivity of $R$.

Proof: all instances of $[\![ \varphi \implies \varphi ]\!]$ are reflexive.

But $[\![ \varphi \implies \varphi ]\!]$ is reflexive.

Exercise: this model also validates $T$. But it is not reflexive!

Valid Shapes and Frame Conditions

A relation $R$ is reflexive if $\forall w \in W, w Rw$. The shape $[\![ \varphi \implies \varphi ]\!]$ is called $T$.

Lemma 11 A frame $\langle W, R \rangle$ validates $T$ iff $R$ is reflexive.

Proof: Assume $R$ is reflexive and $\langle W, R, \vartheta \rangle \not\models [\![ \varphi \implies \varphi ]\!]$ for some $\psi$. Hence, $\vartheta \models \neg \psi$.

Proof: Assume $\langle W, R \rangle$ forces all instances of $[\![ \varphi \implies \varphi ]\!]$ for all $w \in W$. But $[\![ p_0 \implies p_0 ]\!]$ is an instance of $T$ hence $w_0 \models [\![ p_0 \implies p_0 ]\!]$. Contradiction.

The Logic of Reflexive Kripke Frames

Let $\mathfrak{R} \mathfrak{T}$ be the class of all reflexive Kripke frames.

Let $\mathcal{K} \mathcal{T}$ be the class of all reflexive Kripke models.

Let $\mathcal{K} \mathcal{T} = \mathcal{K} + [\![ \varphi \implies \varphi ]\!]$ (shape $T$) as an extra modal axiom.

Define $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$ to mean $\forall \mathcal{M} \in \mathcal{K} \mathcal{T}, \mathcal{M} \models \Gamma \Rightarrow \mathcal{M} \models \varphi$.

Define $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$ to mean there is a derivation of $\varphi$ from $\Gamma$ in $\mathcal{K} \mathcal{T}$.

Soundness: if $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$ then $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$.

Proof: all instances of $T$ are valid in reflexive frames.

Completeness: if $\Gamma \vdash_{\mathcal{K} \mathcal{T}} \varphi$ then $\mathcal{M} \models \varphi$ and $\mathcal{M} \in \mathcal{K} \mathcal{T}$.

Proof: if $\mathcal{M} \models \varphi$ then $\mathcal{M} \models [\![ \varphi \implies \varphi ]\!]$ (sic).

i.e. $T$-instance $[\![ \psi \implies \psi ]\!]$ is valid in $\mathcal{M}$.

$\forall w, v \in W, w Rw$ if $\psi \models [\![ \psi \implies \psi ]\!] \subseteq v$ implies $w Rw$.
Correspondence, Canonicity and Completeness

Normal modal logic $L$ is determined by class of Kripke frames $\mathcal{C}$ if:
$$\forall \varphi, \mathcal{C} \models \varphi \iff \vdash_L \varphi$$
Normal modal logic $L$ is complete if determined by some class of Kripke frames. A normal modal logic is canonical if it is determined by its canonical frame.

A Sahlqvist formula is a formula with a particular shape (too complicated to define here but see Blackburn, de Rijke and Venema)

**Theorem 3** Every Sahlqvist formula $\varphi$ corresponds to some first-order condition on frames, which is effectively computable from $\varphi$.

**Theorem 4** If each axiom $A_i$ is a Sahlqvist formula, then the Hilbert logic $\mathcal{K}A_1A_2\cdots A_n$ is canonical, and is determined by a class of frames which are first-order definable.

**Theorem 5** Given a collection of Sahlqvist axioms $A_1, \cdots, A_k$, the logic $\mathcal{K}A_1A_2\cdots A_k$ is complete wrt the class of frames determined by $A_1 \cdots A_k$.

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Second-Order Aspects of Modal Logics

All of these conditions are first-order definable so it looked like modal logic was just a fragment of first-order logic ...

An $R$-chain is a sequence of distinct worlds $w_0 R w_1 R w_2 \cdots$.

<table>
<thead>
<tr>
<th>Name</th>
<th>Shape</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G$</td>
<td>$[]([\varphi] \rightarrow [\varphi]) \rightarrow [\varphi]$</td>
<td>transitive and no infinite $R$-chains</td>
</tr>
<tr>
<td>$Grz$</td>
<td>$[\varphi]([\varphi] \rightarrow [\varphi]) \rightarrow [\varphi]$</td>
<td>reflexive, transitive and no infinite $R$-chains</td>
</tr>
</tbody>
</table>

The condition “no infinite $R$-chains” is not first-order definable since “finiteness” is not first-order definable. It requires second-order logic, so propositional modal logic is a fragment of second-order logic.

The logic $\mathcal{K}G$ has an interesting interpretation where $[\varphi]$ can be read as “$\varphi$ is provable in Peano Arithmetic”.

These logics are not Sahlqvist.

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Not All First-Order Conditions Are Captured By Shapes

**Theorem 6 (Chagrov)** It is undecidable whether an arbitrary modal formula has a first-order correspondent.

Question: Are there conditions on $R$ not captured by any shape?

Yes: the following conditions cannot be captured by any shape:

- **Irreflexivity**: $\forall w \in W$, not $w R w$
- **Anti-Symmetry**: $\forall u, v \in W, u R v \land v R u \Rightarrow u = v$
- **Asymmetry**: $\forall u, v \in W, u R v \Rightarrow \neg (v R u)$

See Goldblatt for details.

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More Axiom and Frame Correspondences

<table>
<thead>
<tr>
<th>Name</th>
<th>Axiom</th>
<th>Frame Class</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$[]\varphi \rightarrow \varphi$</td>
<td>Reflexive</td>
<td>$\forall w \in W, w R w$</td>
</tr>
<tr>
<td>$D$</td>
<td>$[]\varphi \rightarrow [\varphi]$</td>
<td>Serial</td>
<td>$\forall w \in W, w R w$</td>
</tr>
<tr>
<td>4</td>
<td>$[]\varphi \rightarrow [\varphi]$</td>
<td>Transitive</td>
<td>$\forall w, u R v \Rightarrow u R w$</td>
</tr>
<tr>
<td>5</td>
<td>$[\varphi] [\varphi] \rightarrow [\varphi]$</td>
<td>Euclidean</td>
<td>$\forall w, u R v \Rightarrow v R w$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\varphi \rightarrow [\varphi]$</td>
<td>Symmetric</td>
<td>$\forall u, w \in W, u R v \Rightarrow v R u$</td>
</tr>
<tr>
<td>$A_1$</td>
<td>$[\varphi] [\varphi] \rightarrow [\varphi]$</td>
<td>Weakly-Functional</td>
<td>$\forall u, v, w \in W, u R v \Rightarrow v = w$</td>
</tr>
<tr>
<td>2</td>
<td>$[\varphi] [\varphi] \rightarrow [\varphi]$</td>
<td>Weakly-Directed</td>
<td>$\forall u, v, w \in W, u R v \Rightarrow \exists x \in W, v R x \land u R x$</td>
</tr>
<tr>
<td>3</td>
<td>$[\varphi] [\varphi] \rightarrow [\varphi]$</td>
<td>Weakly-Linear</td>
<td>$\forall u, v, w \in W, u R v \Rightarrow v R w$ or $w R v$ or $w = v$</td>
</tr>
</tbody>
</table>

Let $KA_1A_2\cdots A_n = K + A_1 + A_2 + \cdots + A_n$. (any $A_i$s from above)

**Theorem 2** $\Gamma \vdash KA_1A_2\cdots A_n \varphi$ if $\Gamma \models KA_1A_2\cdots A_n \varphi$

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Finding derivations in Hilbert Calculi is cumbersome: instead of is not always applicable because modal logics do not have a questions can be answered via refinements of canonical models filtrations.

**Shapes Not Captured By Any Kripke Frame Class**

Consider logic $\mathbf{K} \mathbf{I}I$ where $I$ is the axiom schema $[[[] \varphi \leftrightarrow \varphi]] \rightarrow [[\varphi]]$.

**Theorem 7 (Boolos and Sambin)** The logic $\mathbf{K} \mathbf{I}I$ is not determined by any class of Kripke frames.


Incompleteness first found in modal logic by S K Thomason in 1972. Beware, there is also a R H Thomason in modal logic literature.

Can regain a general frame correspondence by using general frames instead of Kripke frames: see Kracht.

Kracht shows how to compute modal Sahlqvist formulae from first-order formulae.

SCAN Algorithm of Dov Gabbay and Hans Juergen Ohlbach automatically computes first-order equivalents via the web.

**Sub-Normal Mono-Modal Logics**

Hilbert Calculus $\mathbf{S} = \mathbf{PC}$ plus modal axioms (not $\mathbf{K}$)

(Id) $\Gamma \vdash \varphi \in \Gamma$

(Ax) $\Gamma \vdash \varphi$ is an instance of an axiom schema

(MP) $\Gamma \vdash \varphi, \Gamma \vdash \psi \rightarrow \psi$

(Mon) $\Gamma \vdash \varphi \rightarrow \psi$

(Nec) no rule

$\Gamma \vdash \varphi : \text{iff there is a derivation of } \varphi \text{ from } \Gamma \text{ in } \mathbf{S}$.

Such modal logics are called “sub-normal”.

$\Gamma \models \varphi : \text{ needs Kripke models } (W, Q, R, \vartheta) \text{ where: } W \text{ is a set of “normal” worlds and } \vartheta \text{ behaves as usual, and } Q \text{ is a set of “queer” or “non-normal” worlds where } \vartheta(w_q, i) = t \text{ for all } \varphi \text{ and all } w_q \in Q \text{ by definition. Then } (\text{Nec})$ fails since $M \models \varphi \models \mathbf{M} \models \varphi \text{ i.e. every non-normal world makes } [[\varphi]] \text{ false.}$

Applications in logics for agents: $\models \varphi \Rightarrow [[\varphi]] \text{ says that “if } \varphi \text{ is valid, then } \varphi \text{ is known”, but agents may not be omniscient, hence want to go “sub-normal”}$.

**Regaining Expressive Power Via Nominals**

Atomic Formulae: $p ::= p_0 \mid p_1 \mid p_2 \mid \cdots$ \hspace{1cm} (Atm)

Nominals: $i ::= i_0 \mid i_1 \mid i_2 \mid \cdots$ \hspace{1cm} (Nom)

Formulae: $\varphi ::= p \mid i \mid \neg \varphi \mid [\varphi] \mid [\varphi]\varphi \mid \varphi \land \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$ \hspace{1cm} (Fml)

Valuation: $\vartheta(w, i) = t$ at only one world for every nominal $i$

Intuition: $i$ is the name of $w$.

Expressive Power:

Irreflexivity: $\forall w \in W, \text{ not } w R w$

$\quad i \rightarrow \neg [i]i$

Anti-Symmetry: $\forall u, v \in W, u R v \& v R u \Rightarrow u = v$

$\quad i \rightarrow [i]([\neg i]i \rightarrow i)$

Asymmetry: $\forall u, v \in W, u R v \Rightarrow \neg (v R u)$

$\quad i \rightarrow \neg [[\neg i]i$


**Lecture 4: Tableaux Calculi and Decidability**

**Motivation:** Finding derivations in Hilbert Calculi is cumbersome:

$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash \varphi \rightarrow \psi \text{ fails!} \quad \Gamma, \varphi \vdash \psi \iff \Gamma \vdash ([]^0 \varphi \land [1] \varphi \cdots [n] \varphi) \rightarrow \psi$

$\vdash \varphi \quad \vdash [\varphi]$

$\vdash [\varphi]$

Resolution: is not always applicable because modal logics do not have a clausal normal form.

Decidability: questions can be answered via refinements of canonical models called filtrations, but there are better ways ...

For filtrations see Goldblatt.

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Negated Normal Form

NNF: A formula is in negated normal form iff all occurrences of \( \neg \) appear in front of atomic formulae only, and there are no occurrences of \( \rightarrow \).

**Lemma 12** Every formula \( \varphi \) can be rewritten into a formula \( \varphi' \) such that \( \varphi' \) is in negated normal form, the length of \( \varphi' \) is at most polynomially longer than the length of \( \varphi \), and \( \emptyset \models \varphi \leftrightarrow \varphi' \).

Proof: Repeatedly distribute negation over subformulae using the following valid principles:

\[
\begin{align*}
\vdash (\varphi_1 \rightarrow \psi_1) & \leftrightarrow (\neg \varphi_1 \lor \psi_1) \\
\vdash (\neg \varphi \land \psi) & \leftrightarrow (\neg \varphi \land \neg \psi) \\
\vdash (\neg \neg \varphi) & \leftrightarrow \neg \neg \varphi
\end{align*}
\]

Examples: NNF

Example:

\[
\neg (\Box(p_0 \rightarrow p_1)) \\
\neg (\Box(p_0 \rightarrow p_1)) \\
\Box(p_0 \land \neg (\Box p_0 \rightarrow \Box p_1)) \\
\Box(p_0 \land \neg (\Box p_0 \rightarrow \Box p_1))
\]

Tableaux Calculi for Normal Modal Logics

**Static Rules:**

\[
\begin{align*}
\frac{(\land) \varphi \land \psi; X}{\varphi; X} & \quad \frac{(\lor) \varphi \lor \psi; X}{\psi; X}
\end{align*}
\]

**Transitional Rule:**

\[
\frac{\Box \varphi; \Box Z}{\Box \varphi; X} \land \psi \varphi \psi; X Z
\]

**Examples of K-Tableau**

\[
\begin{align*}
\frac{(\land) \Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow p_1)}{(\land) \Box(p_0 \rightarrow p_1) \rightarrow (\Box p_0 \rightarrow p_1)} \\
\frac{(\land) \Box(p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0)}{(\land) \Box(p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0)
\end{align*}
\]

A **K-tableau** for \( Y \) is an inverted finite tree of nodes with:

1. a root node \( \text{nff} \ Y \)
2. and such that all children nodes are obtained from their parent node by instantiating a rule of inference

A **K-tableau** is **closed** if all leaves are instances of (id), else it is **open**.

Examples of K-Tableau

\[
\begin{align*}
\Box(p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0) \\
\Box(p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0 \land \Box p_0)
\end{align*}
\]

There is a closed K-tableau for \( \neg (\Box(p_0 \rightarrow p_1)) \).
Examples of Tableau

\[
\begin{align*}
(id) & \quad \frac{p \land \neg p; x}{\therefore} \\
& \quad \frac{\varphi \land \psi; x}{\varphi \lor \psi; x} \\
& \quad \frac{\varphi; x}{\psi; x} \\
& \quad \frac{\Box \varphi; x}{\Box Z} \\
& \quad \frac{\neg \psi; \Box X}{\Box \forall \psi; \Box \neg \psi \notin Z}
\end{align*}
\]

Some Proof Theory

\[
\begin{align*}
(id) & \quad \frac{p \land \neg p; x}{\therefore} \\
& \quad \frac{\varphi \land \psi; x}{\varphi \lor \psi; x} \\
& \quad \frac{\varphi; x}{\psi; x} \\
& \quad \frac{\Box \varphi; x}{\Box Z} \\
& \quad \frac{\neg \psi; \Box X}{\Box \forall \psi; \Box \neg \psi \notin Z}
\end{align*}
\]

Soundness of Modal Tableaux W.R.T. \(K\)-satisfiability

A multiset of formulae \(Y\) is \(K\)-satisfiable iff there is some Kripke model \(\langle W, R, \varnothing \rangle\) and some \(w \in W\) with \(w \models Y\) i.e. \(\forall \varphi \in Y, w \models \varphi\).

Lemma 17 (id) The multiset \(p, \neg p; x\) is never \(K\)-satisfiable.

Lemma 18 (\(\land\)) If \(\varphi \land \psi; x\) is \(K\)-satisfiable then \(\varphi; \psi; x\) is \(K\)-satisfiable.

Lemma 19 (\(\lor\)) If \(\varphi \lor \psi; x\) is \(K\)-satisfiable then \(\varphi; \psi; x\) is \(K\)-satisfiable or \(\psi; x\) is \(K\)-satisfiable.

Lemma 20 (\(\Box\)) If \(\Box \varphi; x\) is \(K\)-satisfiable then \(\varphi; x\) is \(K\)-satisfiable.

Suppose \(\Box \varphi; x\) is \(K\)-satisfiable.

- i.e. exists Kripke model \(\langle W, R, \varnothing \rangle\) and some \(w \in W\) with \(w \models \Box \varphi; x\)
- i.e. exists Kripke model \(\langle W, R, \varnothing \rangle\) and some \(v \in W\) with \(wRv\) and \(v \models \varphi\)
- i.e. \(v \models \varphi\) and \(v \models \Box x\) i.e. \(v \models \Box \varphi; x\)
- i.e. exists Kripke model \(\langle W, R, \varnothing \rangle\) and some \(v \in W\) with \(v \models \varphi; x\)

Soundness of Modal Tableaux

Theorem 8 If there is a closed \(K\)-tableau for \(Y\) then \(Y\) is not \(K\)-satisfiable.

Proof: Suppose there is a closed \(K\)-tableau for \(\neg \varphi; x\). Proceed by induction on length of \(K\)-tableau, recall that \(\models \Box \Box Y \leftrightarrow (\Box \Box Y)\).

Ind. Hyp.: Theorem holds for all derivations of length less than some \(k > 0\).

Ind. Step: Then \(\neg \varphi; x\) has a closed \(K\)-tableau of length \(k\). Top-most rule?

(\(\Box\)): So the top-most rule application is an instance of the (\(\Box\))-rule. \(\varphi; x\) has closed \(K\)-tableau By IH, \(\varphi; x\) is not \(K\)-satisfiable.

Lemma 20: if \(\Box \varphi; x\) is \(K\)-satisfiable then \(\varphi; x\) is \(K\)-satisfiable. Hence \(\Box \varphi; x\) cannot be \(K\)-satisfiable.

Corollary 2: If \(\Box \neg \varphi\) has a closed \(K\)-tableau then \(\models \varphi\)
### Downward Saturated Or Hintikka Sets

A set $Y$ is downward-saturated or an Hintikka set iff:

- $\neg \phi \in Y \Rightarrow \phi \in Y$
- $\forall \phi \land \psi \in Y \Rightarrow \phi \in Y$ and $\psi \in Y$
- $\forall \phi \lor \psi \in Y \Rightarrow \phi \in Y$ or $\psi \in Y$
- $\forall \phi \Rightarrow \psi \in Y \Rightarrow \phi \notin Y$ or $\psi \in Y$

Downward-saturated set is consistent if it does not contain $\{\phi, \neg \phi\}$, for any $\phi$.

Don't need maximality: it is not demanded that $\forall \phi. \phi \in Y$ or $\neg \phi \in Y$. (Hintikka)

### Completeness and Decidability

#### Lemma 21 (Hintikka)

If there is a $K$-model-graph $\langle W, < \rangle$ for set $Y$ then $Y$ is $K$-satisfiable.

Proof: Let $\langle W, R, \varphi \rangle$ be the model where $R = <$ and $\varphi(w, p) = t$ iff $p \in w$. By induction on the length of a formula $\varphi$, show that $\varphi(w, \varphi) = t$ iff $\varphi \in w$. Since $Y \subseteq w_0$ we have $w_0 \vdash Y$.

### Model Graphs

A $K$-model-graph for set $Y$ is a pair $\langle W, < \rangle$ where $W$ is a non-empty set of downward-saturated and consistent sets, some $w_0 \in W$ contains $Y$, and $<$ is a binary relation over $W$ such that for all $w$:

- $\langle \rangle : \langle \varphi \in w \Rightarrow (\exists v \in W, w < v \land \varphi \in v) \rangle$
- $[\varphi] : \langle \varphi \in w \Rightarrow (\forall v \in W, w < v \Rightarrow \varphi \in v) \rangle$.

#### Lemma 22

If every $K$-tableau for $Y$ is open, then $Y$ can be extended into a downward-saturated and consistent $Y^*$ so every $K$-tableau for $Y^*$ is also open.

Proof: Suppose no $K$-tableau for $Y$ closes. Now consider the following systematically constructed $K$-tableau.

Stage 0: Let $w_0 = Y$.

Stage 1: Apply static rules giving finite open branch of nodes $w_0, w_1, \ldots, w_k$.

Let $Y^*$ be the multiset-union of $w_0, \ldots, w_k$.

Claim: $Y^*$ is downward-saturated (obvious) and consistent, and $Y \subseteq Y^*$.

By Contraction Lemma 16, we know $\varphi; X$ has (no) closed $K$-tableau iff $\varphi; \varphi; X$ has (no) closed $K$-tableau. (adding copies cannot affect closure)

Tableau for $Y^*$ cannot close since construction of $Y^*$ just adds back the principal formulae of each static rule application. can treat $Y^*$ as a set!
Decidability and Analytic Superformula Property

Subformula property: the nodes (sets) of a $K$-tableau for $Y$ (i.e. $\text{nnf } Y$) only contain formulae from $\text{nnf } Y$.

Subformula property will hold if all rules simply break down formulae or copy formulae across.

Analytic superformula property: the nodes (sets) of a $K$-tableau for $Y$ (i.e. $\text{nnf } Y$) only contain formulae from a finite set $Y'$ computable from $\text{nnf } Y$ (but possibly larger than $\text{nnf } Y$).

Analytic superformula property will hold if all rules that build up formulae cannot be applied ad infinitum.

The main skill in tableau calculi is to invent rules with the subformula property or the analytic superformula property!

What About Logical Consequence $\Gamma \models \varphi$?

Intuition: the ($\emptyset$) rule captures the semantics of $\emptyset$ by creating an $R$-successor.

Recall that $\Gamma \models \varphi$ means $(\forall M. M \models \Gamma \Rightarrow M \models \varphi)$

Q: how to ensure that every world created by our tableau method forces $\Gamma$?

Write $\Gamma \vdash \varphi$ iff there is a closed $K$-tableau for $(\Gamma; \neg \varphi)$ i.e. $\text{nnf } (\Gamma; \neg \varphi)$

Note: the root world must now force $\Gamma$ and make $\varphi$ false.

Want: $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$ and $\Gamma \models \varphi \Rightarrow \Gamma \not\models \varphi$

Soundness: $\Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi$ iff there is a closed $K$-tableau for $(\Gamma; \neg \varphi)$

iff $(\Gamma; \neg \varphi)$ is not $K$-satisfiable iff not $(\exists M. \exists w, w \models (\Gamma; \neg \varphi))$

iff $(\forall M. \forall w. w \not\models (\Gamma; \neg \varphi))$

iff $(\forall M. \forall w. w \vdash (\Gamma \rightarrow \varphi))$

iff $(\forall M. M \models ((\forall \Gamma \rightarrow \varphi)) \Rightarrow (\forall M. M \models \Gamma \Rightarrow M \models \varphi)$

Completeness W.R.T. $\mathcal{K}$-Satisfiability

Theorem 9 If there is no closed $K$-tableau for $Y$ then $Y$ is $\mathcal{K}$-satisfiable.

Proof: Suppose every $K$-tableau for $Y$ is open.

Use Lemma 23 to construct a $K$-model-graph $\langle W, \vartheta \rangle$ for $Y$.

For all $w \in W$, let $\vartheta(w, p) = t$ iff $p \in w$.

Then $\langle W, \vartheta, \vartheta \rangle$ contains a world $w_0$ with $w_0 \models Y$ by Hintikka’s Lemma 21.

Corollary 3 If there is no closed $K$-tableau for $\neg \varphi$ then $\not\models \varphi$.

Corollary 4 There is a closed $K$-tableau for $Y$ iff $Y$ is not $\mathcal{K}$-satisfiable.

Corollary 5 There is a closed $K$-tableau for $\neg \varphi$ iff $\varphi$ is $K$-valid.

What About Logical Consequence: a concrete example

Write $\Gamma \vdash \varphi :$ iff there is a closed $K$-tableau for $(\Gamma; \neg \varphi)$ i.e. $\text{nnf } (\Gamma; \neg \varphi)$

Want Completeness: $\Gamma \vdash \varphi \Rightarrow \exists M. M \models \Gamma \land M \not\models \varphi$

Consider: $\Gamma := \{p_0\}$ and $\varphi := \{\}\not\models p_1$.

Then $\text{nnf } (\Gamma; \neg \varphi)$ has only one (open) $K$-tableau:

\[
\begin{align*}
(\Gamma; \neg \varphi) & \quad (p_0; \neg \{p_1\}) \\
(p_0; \emptyset) \not\models \varphi & \quad \text{nnf} \\
\not\models \varphi_1 & \quad (\emptyset)
\end{align*}
\]

$w_0 = \{p_0, \not\models p_1\} \quad w_1 = \{\not\models p_1\} \quad w_0 R w_1$

Problem: although $w_0 \models \Gamma$, we don’t have $w_1 \models \Gamma$. So $M \not\models \varphi$ but $M \models \Gamma$.

If only we could make $w_1$ force $\Gamma$ too ...
Regaining Completeness WRT Logical Consequence

Change ($\emptyset$) rule from ($\emptyset$) \[ \frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z \] to:

Transitional Rule: ($\emptyset$) $\Gamma$ \[ \frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z \quad (R\text{-successor forces } \Gamma) \]

Semantic reading:
if numerator is $L$-satisfiable in a model that forces $\Gamma$
then some denominator is $L$-satisfiable in a model that forces $\Gamma$

Stage 2: For each $i = 1 \ldots n$ apply: ($\emptyset$) $\Gamma$ \[ \frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z \]

By completeness: $\Gamma \vdash^T \varphi$:
iff $(\exists M, \exists w, M \models \Gamma \& w \models (\Gamma ; \, \neg \varphi))$

iff $(\exists M, M \models \Gamma \& \exists w \models (\Gamma ; \, \neg \varphi))$

But there is a slight problem ...

Regaining Decidability

Problem: $K\text{-tableau can now loop for ever}$: $\Gamma := \{\emptyset p_0\}$, and $\varphi := p_1$:

\[ (\emptyset ; \, \neg \varphi) \]
\[ \cdot \cdot \cdot \cdot \cdot (\text{nnf}) \]
\[ (\emptyset p_0 ; \, \neg p_1) \]
\[ (\emptyset \Gamma) \]
\[ (p_0 ; \, (\emptyset p_0)) \]
\[ (\emptyset p_0) \]
\[ (\emptyset \Gamma) \]
\[ \cdot \cdot \cdot \cdot \cdot \]

Solution: if we ever see a repeated node, just add a $<$-edge back to previous copy on path from current node to root.

Other Normal Modal Logics

$K T$: Static Rules: ($id$), ($\land$), ($\lor$), plus ($T$) $\frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z$ (unstar all $[]$-formulae)

Transitional Rule: ($\emptyset$) $\Gamma$ $\frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z$ (unstar all $[]$-formulae)

$K 4$: Static Rules: ($id$), ($\land$), ($\lor$)

Transitional Rule: ($\emptyset$) $\Gamma$ $\frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z$

$K T 4$: Static Rules: ($id$), ($\land$), ($\lor$), ($T$)

Transitional Rule: ($\emptyset$) $\Gamma$ $\frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z$ (unstar all $[]$-formulae)

Examples of $K T$-Tableau

$K T$: Static Rules: ($id$), ($\land$), ($\lor$), plus ($T$) $\frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z$ (unstar all $[]$-formulae)

Transitional Rule: ($\emptyset$) $\Gamma$ $\frac{\{\varphi; \, [X; \, Z] \}}{\varphi; \, X} \forall \psi, \, [\psi] \notin Z$ (unstar all $[]$-formulae)

There is a closed $K T$-tableau for $\neg (\{p_0 \rightarrow p_0\}$ i.e. $\emptyset \vdash^T \{p_0 \rightarrow p_0\}$. Starring stops infinite sequence of $T$-rule applications.
**Examples of $K4$-Tableau**

$K4$: Static Rules: $(id), (\land), (\lor)$

Transitional Rule: $(\langle \rangle \land)$

$\vdash \psi; X; [[X; Z] \land \forall \psi; [], \psi \notin Z]

$\vdash ([]p_0 \rightarrow [[]p_0])

$([]p_0) \land (\langle \rangle \neg p_0)

\vdash ([]p_0; [[]p_0])

$([]p_0; [[]p_0])

$\vdash ([]p_0; [[]p_0])

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Lecture 5: Tense and Temporal Logics

Tense Logics: interpret $[t] \varphi$ as "$\varphi$ is true always in the future".

$W$ represents moments of time
$R$ captures the flow of time

Temporal Logics: similar interpretation but use a more expressive binary modality $\mathcal{U} \psi$ to capture "$\varphi$ is true all time points from now until $\psi$ becomes true".

Shall look at Syntax, Semantics, Hilbert and Tableau Calculi.

Tense Logics: Syntax and Semantics

Atomic Formulae:

Formulae:

Boolean connectives interpreted as for modal logic.

Given some Kripke model $\langle W, R, \vartheta \rangle$ and some $w \in W$, we compute the truth value of a non-atomic formula by recursion on its shape:

Example: If $W = \{w_0, w_1, w_2\}$ and $R = \{(w_0, w_1), (w_0, w_2)\}$ and $\vartheta(w_1, p_3) = t$ then $\langle W, R, \vartheta \rangle$ is a Kripke model as pictured below:

Let $\mathcal{K}_t = \mathcal{K}$ be the class of all Kripke Tense frames.

Hilbert Calculus for Modal Logic $K_t$

Axiom Schemata: Axioms for $PC$ plus:

$K[F]$: $[F](\varphi \rightarrow \psi) \rightarrow ([F] \varphi \rightarrow [F] \psi)$

$K[P]$: $[P](\varphi \rightarrow \psi) \rightarrow ([P] \varphi \rightarrow [P] \psi)$

$FP$: $\varphi \rightarrow [F]([P] \varphi)$

$PF$: $\varphi \rightarrow [P](F) \varphi$

Rules of Inference: (Ax) $\varphi$ is an instance of an axiom schema

(id) $\varphi \in \Gamma$

(MP) $\Gamma \vdash K_t \varphi \quad \Gamma \vdash K_t \psi \quad \Gamma \vdash [F] \varphi \rightarrow \psi$

(Nec[F]) $\Gamma \vdash K_t \varphi \quad \Gamma \vdash [F] \varphi$

(Nec[P]) $\Gamma \vdash K_t \varphi \quad \Gamma \vdash [P] \varphi$

Soundness and Completeness:

$\Gamma \vdash K_t A_1, A_2, \ldots, A_n \varphi$ iff $\Gamma \models K_t A_1, A_2, \ldots, A_n \varphi$
Different Models of Time

Arbitrary Time: $K_t$

Reflexive Time: $\varphi \rightarrow (F)\varphi$

Transitive Time: $(F)(F)\varphi \rightarrow (F)\varphi$

Dense Time: $(F)\varphi \rightarrow (F)(F)\varphi$

Never Ending Time: $[F]\varphi \rightarrow (F)\varphi$

Backward Linear: $(F)[F]\varphi \rightarrow (F)\varphi \lor (F)\varphi \lor \varphi$

Forward Linear: $(F)[F]\varphi \rightarrow (F)\varphi \lor (F)\varphi \lor \varphi$

Discrete $\langle Z, < \rangle$, Rational $\langle Q, < \rangle$, Real $\langle R, < \rangle$ linear and non-reflexive models of time also possible: see Goldblatt.

Tableau Calculi also exist but require even more complex loop detection often called "dynamic blocking".

PLTL: Propositional Linear Temporal Logic

Atomic Formulae: $p ::= p_0 | p_1 | p_2 | \cdots$

Formulae: $\varphi ::= p | \neg \varphi | \bigoplus \varphi | (F)\varphi | (F)\varphi \bigcirc \psi | \varphi \land \varphi | \varphi \lor \varphi | \varphi \rightarrow \varphi$

Boolean connectives interpreted as for modal logic.

Linear Time Kripke Model: $\langle S, \sigma, \vartheta \rangle$

$S$: non-empty set of states

$\sigma$: $\mathbb{N} \rightarrow S$ enumerates $S$ as sequence $\sigma_0, \sigma_1, \cdots$ with repetitions when $S$ finite

$\vartheta$: $S \times \text{Atm} \rightarrow \{t, f\}$

Semantics of PLTL

$\vartheta(s_i, \bigoplus \varphi) = \begin{cases} t & \text{if } \vartheta(s_{i+1}, \varphi) = t \\ f & \text{otherwise} \end{cases}$

$\vartheta(s_i, (F)\varphi) = \begin{cases} t & \text{if } \vartheta(s_j, \varphi) = t \text{ for some } j \geq i \\ f & \text{otherwise} \end{cases}$

$\vartheta(s_i, [F]\varphi) = \begin{cases} t & \text{if } \vartheta(s_j, \varphi) = t \text{ for all } j \geq i \\ f & \text{otherwise} \end{cases}$

$\vartheta(s_i, \bigcirc \psi) = \begin{cases} t & \text{if } \exists k \geq i, \vartheta(s_k, \psi) = t \text{ & } \forall j, i \leq j < k \Rightarrow \vartheta(s_j, \varphi) = t \\ f & \text{otherwise} \end{cases}$

Note: when $k \neq i$, $s_k$ is first state after $s_i$ where $q$ is true.
Hilbert Calculus for PLTL

Axiom Schemata: axioms for $\text{PC plus}$

$K^F$: $[F](\varphi \rightarrow \psi) \rightarrow ([F]\varphi \rightarrow [F]\psi)$

$K\oplus$: $\oplus(\varphi \rightarrow \psi) \rightarrow (\oplus\varphi \rightarrow \oplus\psi)$

$\text{Fun}$: $\ominus\psi \leftrightarrow \lnot\ominus\psi$

$\text{Mix}$: $[F]\varphi \rightarrow (\varphi \wedge \oplus[F]\varphi)$

$\text{Ind}$: $[F](\varphi \rightarrow \oplus\varphi) \rightarrow (\varphi \rightarrow [F]\varphi)$

$\cal U_1$: $(\varphi \cal U\psi) \rightarrow \langle F\rangle\psi$

$\cal U_2$: $(\varphi \cal U\psi) \leftrightarrow \psi \lor (\lnot\psi \land \varphi \land \oplus(\varphi \cal U\psi))$

Rules: $(\text{Id})$, $(\text{Ax})$, $\text{M P}$ and $(\text{Nec}[F])$ and $(\text{Nec}\oplus)$

Tableau Calculus for PLTL

Stage 0: put $w_0 = Y$

Stage 1: repeatedly apply usual $(\land)$ and $(\lor)$ rules together with the following to obtain a downward-saturated node $w^n_0$ in which every non-atomic formula is of the form $\oplus\varphi$: $\lnot\oplus\varphi \rightarrow \ominus\lnot\varphi$ $[F]\varphi \rightarrow (\varphi \land \oplus[F]\varphi)$ $\langle F\rangle\varphi \rightarrow (\varphi \lor \langle F\rangle\varphi)$ $\cal U_2: (\varphi \cal U\psi) \rightarrow \psi \lor (\lnot\psi \land \varphi \land \oplus(\varphi \cal U\psi))$

Stage 3: Current node is now of the form $\oplus X$: $Z$ where $Z$ contains only atomic formulae and their negations. So create a $\ominus$-successor $w_1$ containing $X$.

Stage 4: Saturate $w_1$ via Stage 1 to get $w^n_1$ and add $w^n_0Rw^n_1$ if $w^n_1$ is new, else add $w^n_0Rw^n$ for the node $w^n$ which already replicates $w^n_1$.

Stage 5: If $w^n_1$ is new then repeat and so on until no new nodes turn up giving a possibly cyclic graph.

Tableau Method for PLTL: Pass 2

An eventuality is a formula $\langle F\rangle\varphi$.

Delete all nodes that contain a $p$ and $\neg p$ pair.

Delete all nodes who now do not have an $R$-successor.

Stage 3: starting from $w^n_0$, check every path in the graph to make sure that $\langle F\rangle\varphi \in w^n_1 \Rightarrow \exists w^n_0, w^n_1R_0, \ldots R_nw^n_1$ & $\varphi \in w^n_1$.

If all eventualities are fulfilled on a single path then $Y$ is $\text{PLTL}$-satisfiable, otherwise it is not.

Note: all eventualities on a path must be fulfilled on that path!
Extensions to Other Temporal Logics

Extends to temporal logics with a $\Box$ operator and a $\varphi \mathcal{S} \psi$ operator.

Extends to branching-temporal logics.

Usually requires space exponential in size of $Y$ since we cannot follow each branch individually but must construct the whole state space for $Y$ first.

Two pass nature compiles away the induction hypothesis.

Further Reading

G E Hughes and M J Cresswell A New Introduction to Modal Logic Routledge, 1996

Logics of Time and Computation R. I. Goldblatt CSLI Lecture Notes Number 7, Center for the Study of Language and Information, Stanford, 1987

Modal Logic P Blackburn, M de Rijke and Y Venema Cambridge University Press


