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Melbourne University Postgraduate Seminar: "Goedel, Turing, Chaitin"

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Sequents, Automated Reasoning and Nonclassical Logics

1999
Aspects of Formal Logic

Aim: to distinguish good deductions from the bad ones

Syntax: defines the legal expressions of a formal language

Semantics: defines the meaning of the syntactic symbols

Calculus: a purely syntactic method for identifying good and bad deductions

Example: Classical Propositional Logic is a formalisation of the linguistic concepts of "and", "or", "not" and "if ... then ..."
Classical Propositional Logic

Syntax:

Every proposition \( p \) is a formula, and if \( A \) and \( B \) are formulae then

- \( \neg A \), \( A \lor B \), \( A \land B \), \( A \rightarrow B \), \( A \leftrightarrow B \)

Formulae:

Example: Proposition \( p \) is a formula, and if \( A \) and \( B \) are formulae then

- \( \neg p \), \( p_1, p_2, p_3, \ldots \)

Semantics:

Each interpretation assigns either \( \text{true} \) or \( \text{false} \), but not both, to each proposition. Each proposition \( p \) is either \( \text{true} \) or \( \text{false} \), but not both.

Example: Because Isaac interprets \( p \) as "today is Friday," \( f = (d) \land t = (d) \land \). Example: Because Raj interprets \( p \) as "today is Tuesday," \( f = (d) \land t = (d) \land \).

Intuition:

Each proposition \( p \) is either \( \text{true} \) or \( \text{false} \), but not both.

Example: Because Isaac interprets \( p \) as "today is Friday," \( f = (d) \land t = (d) \land \).
Classical Propositional Logic: Semantics

\[ f = (\exists \, x) \land \text{and } \tau = (\forall \, x) \land \iff f = (\exists \, x \land \forall \, x) \land \]

(\exists \, x... then ... then)

\[ \tau = (\exists \, x) \land \text{or } f = (\forall \, x) \land \iff \tau = (\exists \, x \lor \forall \, x) \land \]

(\exists \, x... or... or)

\[ f = (\exists \, x) \land \text{and } f = (\forall \, x) \land \iff f = (\exists \, x \land \forall \, x) \land \]

(\exists \, x... and... and)

\[ \tau = (\exists \, x) \land \text{or } \tau = (\forall \, x) \land \iff \tau = (\exists \, x \lor \forall \, x) \land \]

(\exists \, x... or... or)

\[ \tau = (\forall \, x) \land \iff f = (\forall \, x) \land \iff \tau = (\forall \, x \land \forall \, x) \land \]

(not)

\[ f = (\forall \, x) \land \iff \tau = (\forall \, x) \land \iff \tau = (\forall \, x \lor \forall \, x) \land \]

(\forall... or... or)...
Interpretations. How to recognise valid formulae without trying them all?

**Problem:** A formula built from \( n \) different propositions has \( 2^n \) different interpretations.

**Examples of valid formulae:**

\[
\begin{align*}
f &= (\top d) \land \top d \land f = (\top d \subset 0d) \land : \top d \subset 0d \\
t &= (\top d) \land \top d \land t = (\top d \subset 0d) \land : \top d \subset 0d \\
\top d \land 0d & \quad \top d \subset 0d
\end{align*}
\]

**Examples of invalid formulae:**

\[
\begin{align*}
f &= (0d) \land \top d \land f = (0d \subset 0d) \land : 0d \subset 0d \\
t &= (0d) \land \top d \land t = (0d \subset 0d) \land : 0d \subset 0d \\
(0d \lor 0d) & \quad 0d \land 0d \quad 0d \subset 0d
\end{align*}
\]

**Validity:** A formula is valid \( \text{iff} \) it is always true in every interpretation.

**Universal Truths:**
Logical Calculi for Automated Reasoning and Non-classical Logics

Algorithmic: Search for derivation should terminate if logic is decidable.

Algorithmic: If calculus derives formula \( f = \bot \) with \( A \) is \( \forall \) \( A \) is not valid.

Algorithmic: Should provide some guidance for finding derivation when asked.

Algorithmic: Calculation cannot derive \( A \) then \( A \) must be invalid.

Complete: If formula \( A \) is valid then calculus must derive \( A \). Equivalently, if calculus cannot derive \( A \) then \( A \) must be valid.

Correct: If calculus derives formula \( A \) then \( A \) must be valid.

Interpretations are subjective. Since interpretations are subjective, must not make reference to interpretations.

Purly syntactic: Must not make references to interpretations, since interpretations are subjective.
Gentzen's Sequent Calculi

**Extrasymbols:**

- $\vdash$
- $\models$
- $\forall$
- $\exists$
- $\rightarrow$
- $\leftrightarrow$
- $\neg$
- $\land$
- $\lor$
- $\Rightarrow$
- $\Leftrightarrow$
- $\forall$
- $\exists$
- $\vdash$
- $\models$
- $\land$
- $\lor$
- $\Rightarrow$
- $\Leftrightarrow$
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- $\models$
- $\land$
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- $\Rightarrow$
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- $\models$
- $\land$
- $\lor$
- $\Rightarrow$
- $\Leftrightarrow$
- $\vdash$
Each rule introduces the principal formula into the conclusion from the side formulae in the premises.
Involving no logical connectives, they only change the shape or structure of a sequent.

Structural Rules: LK 1.0: Structural Rules

<table>
<thead>
<tr>
<th>Structural Rule</th>
<th>LK 1.0: Structural Rules</th>
</tr>
</thead>
</table>
| **Composition** | \[
\begin{array}{c}
\vdash A \\
A, A \vdash \Box \\
\vdash A, A, \Box \\
\end{array}
\] |
| **Contraction** | \[
\begin{array}{c}
\vdash \Box, A, A \\
A, A \vdash \Box \\
A, A, \Box \vdash \Box \\
\end{array}
\] |
| **Weakening** | \[
\begin{array}{c}
\vdash A \\
\vdash \Box \\
\vdash \Box, A \\
\end{array}
\] |
| **Exchange** | \[
\begin{array}{c}
\vdash B, A \\
B, A \vdash \Box \\
A, \Box \vdash B \\
\end{array}
\] |
Example: \( b \leftarrow (b \subset d) \vee d \) is derivable in LK 1.0 (\text{green principal formulae})
Evaluating LK 1.0: correctness

Sequents as formulae:

If \( \varphi \) is \( \vdash \varphi \) (conclusion) \( \varphi \) is derivable then \( \vdash \varphi \) is valid.

\[
\text{else if } \varphi \text{ is derivable then } \varphi \text{ is valid.}
\]

Correctness:
If \( \varphi \) is derivable then \( \varphi \) is valid.

Proof:
By induction on the length \( l \) of the longest branch in the given derivation.

Induction Step:
For every rule, show that if the premises of the rule translates into valid formulae, then so does the conclusion.

Base Case, \( l = 0 \):
Derivation translates into valid formulae.

Let \( \varphi \) be the formula \( \varphi \) and \( \varphi \) is derivable then \( \varphi \) is valid.

Evaulating LK 1.0: correctness
Completeness:
If \( \sigma \) is valid then \( \text{sequent } \sigma \) is derivable in LK 1.0.

Proof
Defer for now.

Completeness 2: If \( \text{sequent } \sigma \) is not derivable in LK 1.0 then \( \sigma \) is not valid.

Completeness:
If \( \sigma \) is valid then \( \text{sequent } \sigma \) is derivable in LK 1.0.
Evaluating LK 1.0: algorithmic nature

LK 1.0 recognises good deductions since derivable means valid.

Suppose we want to find a derivation for \( \neg \top \). We know:

\[
b \leftarrow (b \subset d) \lor d
\]

This sequent must be obtained from the other sequents using the rules of LK 1.0, so derivation must look like:

it must end with the sequent:

\[
b \leftarrow (b \subset d) \lor d
\]

Use the rules of LK backwards: choices of rules and principal formul\(\text{e} \)s give a search procedure.

unknown sequents

\[
\begin{align*}
(b \subset d) \lor d & \quad \text{ (i) } \\
& \quad \text{ b } \leftarrow (b \subset d) \lor d
\end{align*}
\]
Can we eliminate them without changing the class of derivable formulæ?

\[ (\exists x - (\forall x - (\forall x - (\forall x - \neg (\forall x - (\forall x - \neg \forall x )) )) )) \]

We cannot tell in advance when and which structural rules are essential:

Recall that we have seen a derivation for this sequent on slide 10.

\[ (\forall v) \quad \frac{b \leftarrow (b \subseteq d) \lor d}{b \leftarrow b \subseteq d} \]

\[ (\forall v) \quad \frac{b \leftarrow b \subseteq d}{b \leftarrow b \quad d \leftarrow} \]

Failed backward searches for derivations in LK 1.0.

Examples
Algorithmic Improvements: LK 2.0

Immaterial, no structural rules except (cut), things that have changed are in green.

LK 2.0: \( \vdash \)\( \neg \), \( \neg \) are now sets of formulae so order and number of occurrences.

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Evaluating LK 2.0: Algorithmic Nature

Example:

\[
\begin{align*}
\neg & \vdash \bot & \text{(cut)} \\
\bot & \vdash \neg A, \bot, \neg A \vdash \neg A & \text{Guidance: Gives better guidance since less structural rules}
\end{align*}
\]

\[
\begin{align*}
\neg & \vdash \neg \neg A & \text{Search: Only problematic rule is the cut rule since it is backward application.}
\end{align*}
\]

\[
\begin{align*}
(b \in c) \lor d & \vdash b & \text{Guidance: Gives better guidance since less structural rules}
\end{align*}
\]

\[
\begin{align*}
\neg & \vdash (b \in c) \lor d & \\
\neg & \vdash (b \in c) \lor d & \text{Example: Seguent is derivable in LK 2.0.}
\end{align*}
\]

\[
\begin{align*}
\neg & \vdash (b \in c) \lor d & \text{Example: Seguent is derivable in LK 2.0.}
\end{align*}
\]
Evaluating LK 2.0

Comparison:
Must be careful when comparing LK 1.0 and LK 2.0 since one uses sequences and the other uses sets, but formulæ are common

Correctness:
If \( \varphi \) is derivable in LK 2.0 then it is also derivable in LK 1.0.

Relative Completeness:
If \( \varphi \) is derivable in LK 1.0 or LK 2.0 then the derivation can be transformed into another derivation of \( \varphi \) in LK 1.0 or LK 2.0 that does not require the cut rule. The cut rule is eliminable (redundant).

Subformula property: In any cut-free derivation of \( \varphi \) in LK 1.0 or LK 2.0, \( \varphi \) is derivable in LK 1.0/2.0. That does not mean that derivable in LK 1.0/2.0 then \( \varphi \) is derivable in LK 1.0/2.0, but if we skipped the completeness of LK 1.0 (but we are skipping the completeness of LK 2.0) we can conclude that it is derivable in LK 1.0.

Hauptsatz: If \( \varphi \) is derivable in LK 1.0 or LK 2.0 then it is also derivable in LK 1.0 or LK 2.0.

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Algorithm: Apply the rules of LK 2.0 backwards (except cut) to \( \forall \) in any order until there are no compound formulae left in the leaf premises.

Evaluating LK 2.0: Decision Procedure

Proof: If sequent \( A \) is not derivable then \( A \) is not valid.

Completeness 2: If the sequent \( \forall \downarrow \) is valid then the formula \( \exists \downarrow \) is valid.

Correctness: If each leaf is an initial sequent (of the form \( \forall \downarrow \)) then the interpretation \( w 
\Rightarrow \) such that

\( v \) and \( \exists d, \ldots, \exists d \)

so this leaf must be of the form: \( \forall \downarrow \) and \( \exists d, \ldots, \exists d \) and right hand side

That is, at least one leaf has no formula common to its left hand side.

\( \Rightarrow \) But the algorithm only stops when there are no compound formulae left and right hand side

Correctness: If each leaf is an initial sequent (of the form \( \forall \downarrow \)) then the interpretation \( v \) and \( \exists d, \ldots, \exists d \) is guaranteed to make \( \top \) valid.

\( v \) and \( \exists d, \ldots, \exists d \)
Evaluating LK 2.0:

Example:

The sequent $b \leftarrow \neg (b \land d)$ is not derivable in LK 2.0.

Reason:

The left leaf $b$ is not an initial sequent.

Valuation:

$f = ((b \leftarrow (b \land b) \lor d)) \land (d) \land (b \land d)\lor (d)$

$(l \lor) \quad b \leftarrow (b \land b) \lor d$

$(l \subset) \quad b \leftarrow d \lor b$

$b \leftarrow d \lor b$

$b \leftarrow d$

$b \leftarrow d$

The sequent $b \leftarrow (b \land d)$ is not derivable in LK 2.0.
many nonclassical logics; most efficient out of the ones shown

LWB: http://www.lwb.unibe.ch/

propositional modal logics

LeanK: http://www.lwb.unibe.ch/de/mod/lean/

full first-order classical logic

LeanTAP: http://www.lwb.unibe.ch/de/lean/tap/

Gentzen's LK Implemented:
Goal: Can we obtain different logics by changing only the structural rules of Gentzen’s calculus?

Substructural logics: LK 3.0

Method: Reinstate structural rules, but remove all aspects of the structural rules, and change the form of the structural rules to remove aspects of other structural rules.

Disadvantages: Gives theoretical insights but makes LK 3.0 less useful for automated deduction.

Gives theoretical insights but makes LK 3.0 less useful for automated deduction.

- Change the form of the structural rules to remove aspects of other structural rules.
- Change the logical rules to respect left to right order and to remove all aspects of weakening and contraction.
- Change the initial sequent from LK 2.0 to remove aspects of weakening.
- Change sets back to sequences.
- Change sets back to sequences.
- Change sets back to sequences.
- Change sets back to sequences.
- Change sets back to sequences.
- Change sets back to sequences.
- Change sets back to sequences.
- Change sets back to sequences.
Sequents consist of finite, possibly empty, sequences of formulae.

Splitting of contexts when rules like $\vdash \land$ are applied upward results in non-determinism: how to split left and right sides into the necessary subsequences like $\vdash \land (\lor)$ (id) are applied upward results in too many possibilities.
LK 3.0: Structural Rules

Composition:

\[
\frac{\Gamma, \varphi \quad \varphi}{\Gamma} \\
\frac{\varphi, \Gamma}{\Gamma}
\]

Contraction:

\[
\frac{\Gamma, \varphi \quad \varphi}{\Gamma} \\
\frac{\varphi, \Gamma}{\Gamma}
\]

Weakening:

\[
\frac{\varphi}{\varphi \quad \varphi}
\]

Exchange:

\[
\frac{\varphi, \psi \quad \Gamma, \psi}{\varphi, \Gamma} \\
\frac{\varphi, \psi \quad \Gamma, \psi}{\varphi, \Gamma}
\]

\[
\frac{\varphi, \psi \quad \Lambda, \psi}{\varphi, \Lambda} \\
\frac{\varphi, \psi \quad \Lambda, \psi}{\varphi, \Lambda}
\]

\[
\frac{\varphi, \psi \quad \Gamma, \psi}{\varphi, \Gamma} \\
\frac{\varphi, \psi \quad \Gamma, \psi}{\varphi, \Gamma}
\]

\[
\frac{\varphi, \psi \quad \Lambda, \psi}{\varphi, \Lambda} \\
\frac{\varphi, \psi \quad \Lambda, \psi}{\varphi, \Lambda}
\]

\[
\frac{\varphi, \psi \quad \Gamma, \psi}{\varphi, \Gamma} \\
\frac{\varphi, \psi \quad \Gamma, \psi}{\varphi, \Gamma}
\]

\[
\frac{\varphi, \psi \quad \Lambda, \psi}{\varphi, \Lambda} \\
\frac{\varphi, \psi \quad \Lambda, \psi}{\varphi, \Lambda}
\]
Objection: Classical logic is too simplistic. Since the validity of $A \rightarrow \neg \neg A$ in classical logic.

Proof by contradiction: Want to prove that $A \rightarrow \neg \neg A$ so assume $B \subset A$ and derive a contradiction. Thus conclude that $A \subset B$.

Example: Let $p_0$ be interpreted as “it is raining” and $\neg p_0$ tells us that “it is raining or it is not raining”. But it does not give us any way to prove which it is.

The only universal truths are the ones for which we have constructive proofs.

Mantra: The only universal truths are the ones for which we have constructive proofs.

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Nonclassical Logics: Intuitionistic Logic
Gentzen's Calculus for Intuitionistic Logic: LJ1.0

LJ1.0: Obtained from LK1.0 by restricting all right hand sides of sequents to contain at most one formula.

LJ1.0: Gentzen's Calculus for Intuitionistic Logic:

LJ2.0: Can also obtain a more refined version which gives a decision procedure but must be more careful.

Disjunction Property: If \( \forall \) \( B \) is derivable in LJ1.0, then \( \forall \) \( A \land B \) is derivable in LJ1.0.

Glivenko: Sequent \( \forall \) \( A \) is not derivable in LJ1.0 \( \iff \) \( A \) is not derivable in LK1.0.

Proof by contradiction fails:

\( \forall \) \( A \) \( \Rightarrow \) \( \neg \neg (\forall \forall \neg B) \) \( \Rightarrow \) \( A \) is derivable in LJ1.0 or \( B \) is derivable in LJ1.0.
Nonclassical Logics: Relevance Logic

Objection:
Even intuitionistic logic is too simplistic since the validity of \( A \land (A \lor B) \) tells us nothing useful. It is somehow relevant to \( B \), the fact that \( A \) is true has no bearing on whether \( B \) is true unless \( A \) is.

Mantra:
The validity of \( B \lor A \) tells us nothing useful.

Resource Reading:
If \( \forall \rightarrow \) is derivable then every formula occurrence in \( \forall \rightarrow \) is derivable for all choices of \( \forall \) and \( \rightarrow \) in a finite amount of time derivable.

Decidability:
Much harder to give decision procedures, some relevance logic are undecidable: no computer program can ever tell whether \( \forall \rightarrow \) is derivable, even intuitionistic logic is too simplistic since the validity of

Sequents:
Can be obtained by deleting weakening from LK 3.0, and adding some other rules.

Semantics:
Much much more complicated, leave for now.
Nonclassical Logics: BCK Logic

Objection: Intuitionistic logic is too simplistic since the validity of \( A \land A \) does not tell us how many times we used the assumption \( A \).

Example: Let \( p \) be interpreted as „I have one dollar“ and \( \neg p \) be interpreted as „I have no dollar“.

Mantra: We must keep track of how many times we use an assumption.

Resource Interpretation: If \( \bigwedge \varphi \) is derivable then every formula occurrence in the sequent must be principal at most once: no contraction.

Decidability: Much harder to give decision procedures, but many are decidable.

Sequent Calculi: Can be obtained from LK 3.0 by deleting Contraction, and adding some other rules.

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Nonclassical Logics: Linear Logic

Objection:
All previous logics are too simplistic. There are also undecidable resource readings. Many resources are hard to give decision procedures, but many are also

Resource Reading:
If \(\phi\) is derivable, then exactly once in these sequents, no contraction or weakening.

Semantics:
Much more complicated, leave for now

Decidability:
Much harder to give decision procedures, but many are also undecidable. Obtained from LK 3.0 by deleting both Contraction and Weakening.

Calculus:
More complicated, leave for now

Decidability:
Much harder to give decision procedures, but many are decidable, some are also undecidable.

The sequent must be principal exactly once: no contraction, no weakening, if \(T \vdash \phi\), derivable then every formula occurrence in

Resource Reading:
If \(T \vdash \phi\) is derivable then exactly once in these sequents, no contraction, no weakening.
Objection:
Not only is the resource reading paramount, but so is the order.

Resource Reading:
If derivable then every formula occurrence in the sequent must be principal exactly once and it must live at the right position in the sequent: no exchange, no weakening, no contraction and adding some other rules.

Decidability is decidable.

Calculus Obtained from LK 3.0 by deleting contraction, weakening, and exchange, and adding some other rules.

Semantics: Much much more complicated, leave for now.

Nonclassical Logics: Lambek Logic
Gentzen Systems for Substructural Logics

Classical Calculus

Intuitionistic Calculus

Relevance Calculus

BCK Calculus

Linear Calculus

Lambek Calculus: singleton on right

Read from bottom to top
Substructural Logics


Substructural Logics

Further Reading


"Gaggle Theory: An Abstraction of Galois Connections and Residuation with Applications to Negation and "

Logic (to appear), perhaps has appeared.


Further Reading


Annual Logic Summer School
Held in December in Canberra
Courses on: Logic, Automated Reasoning, Gödel's Theorems, Philosophical Logics, Artificial Intelligence, and lots more
Fellowships available for students: Deadline 18 October!!
Contact: Diane Kossatz +61-2-6279-8630
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Sequents, Automated Reasoning and Nonclassical Logics 1999